

Components and Exit Times of Brownian Motion in Multiple p -Adic Dimensions

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Joint work with my advisor, David Weisbart: “Components and Exit Times of Brownian Motion in Two or More p -Adic Dimensions.”

<http://arxiv.org/abs/2210.16429>.

Brownian Motion

- The motion of particles suspended in a fluid at dynamic equilibrium
- Statistical Mechanics models (e.g. Einstein, Smoluchowski) as well as Stochastic Process models
- Applications in mathematics, physics, biology, finance...

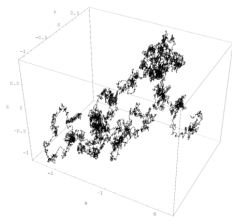


Figure: A realisation of the \mathbb{R}^3 -valued Wiener Process model ¹

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The Classical Wiener Process

$$P(W_t \in \Omega \subset \mathbb{R}) = \int_{\Omega} \frac{1}{\sqrt{2\pi\sigma t}} e^{-\frac{x^2}{2\sigma t}} dx \quad (1)$$

$$u_t = \sigma \Delta u \quad (2)$$

$$P(\vec{W}_t \in \Omega \subset \mathbb{R}^d) = \int_{\Omega} \frac{1}{(2\pi\sigma t)^{d/2}} e^{-\frac{\|\vec{x}\|^2}{2\sigma t}} d\vec{x} \quad (3)$$

$$\vec{W}_t = (W_t^{(1)}, \dots, W_t^{(d)}), \quad W_t^{(i)} \stackrel{iid}{\sim} W_t \quad (4)$$

A Wiener Process Analogue

- (Varadarajan 1997) Stochastic process from fundamental solution to a pseudo-differential analogue of the heat equation for “a vector space of finite dimension over a non-archimedean locally compact division ring”

$$u_t = \sigma (\mathcal{F}^{-1} \mathcal{M}_b \mathcal{F}) u, \quad (\mathcal{M}_b f)(\vec{x}) = \|\vec{x}\|^b f(\vec{x}) \quad (5)$$

- In the concrete case \vec{X}_t valued in \mathbb{Q}_p^d , density given by the fundamental solution

$$\rho_d(t, \vec{x}) = \left(\mathcal{F}^{-1} e^{-\sigma t \|\cdot\|^b} \right) (\vec{x}) \quad (6)$$

$$= \int_{\mathbb{Q}_p^d} \chi(\vec{x} \cdot \vec{y}) e^{-\sigma t \|\vec{y}\|^b} d\vec{y} \quad (7)$$

$$\|\vec{y}\| = \max(|y_1|, \dots, |y_d|)$$

$$|\cdot| \in \{0 < \dots < p^{-1} < p^0 < p^1 < \dots\}$$

Failure of Component Independence

Notation:

$$\vec{X}_t = (X_t^{(1)}, \dots, X_t^{(d)}), \quad \vec{X}_{t,1} = (X_t^{(2)}, \dots, X_t^{(d)})$$

$$B(r, a) = \{x \in \mathbb{Q}_p : |x - a| \leq p^r\}, \quad S(R) = \{x \in \mathbb{Q}_p : |x| = p^R\}$$

For small $t > 0$, integers $r < R$, and $a \in B(R)$:

$$P \left(X_t^{(1)} \in B(r, a) \mid \vec{X}_{t,1} \in \prod_{k=2}^d S(R) \right) \leq p^{r-R} < \frac{1}{p} \quad (8)$$

$$P \left(X_t^{(1)} \in B(r, 0) \right) > \frac{1}{p} \quad (9)$$

Characterizing the Components

Theorem (*R.R. & D. Weisbart*)

- For any positive real number t , the components of \vec{X}_t are not independent.
- The component processes $X_t^{(i)}$ are identically distributed with the same diffusion parameters as \vec{X}_t .

A Key Oscillatory Integral

$$\rho^{(1)}(t, x) = \int_{\mathbb{Q}_p^{d-1}} \rho_d(t, (x, \vec{x}_1)) d\vec{x}_1 \quad (10)$$

$$= \int_{\mathbb{Q}_p^{d-1}} \int_{\mathbb{Q}_p^d} \chi((x, \vec{x}_1) \cdot \vec{y}) e^{-\sigma t \|\vec{y}\|^b} d\vec{y} d\vec{x}_1 \quad (11)$$

$$= \int_{\mathbb{Q}_p} \chi(x \cdot y) e^{-\sigma t |y|^b} dy = \rho_1(t, x) \quad (12)$$

Exit Time Analysis

- Let $Y_t^{(i)}$ be iid with one-dimensional density ρ_1 , and define

$$\vec{Y}_t = \left(Y_t^{(1)}, \dots, Y_t^{(d)} \right)$$

- Extending (Weisbart 2021), set

$$\alpha_d = 1 - \frac{p^b - 1}{p^{b+d} - 1},$$

and for any \mathbb{Q}_p^d -valued stochastic process \vec{Z} define

$$\left\{ \vec{Z} \right\}_T = \sup_{0 \leq t \leq T} \|\vec{Z}_t\|$$

- First exit probabilities are the complement of the survival probabilities:

$$P \left(\left\{ \vec{Y} \right\}_T \leq p^R \right) = e^{-d\sigma\alpha_1 T p^{-Rb}} \xrightarrow{d \rightarrow \infty} 0 \quad (13)$$

$$P \left(\left\{ \vec{X} \right\}_T \leq p^R \right) = e^{-\sigma\alpha_d T p^{-Rb}} \xrightarrow{d \rightarrow \infty} e^{-\sigma T p^{-Rb}} \quad (14)$$

Some Future Directions

- Generalizing to the setting of (Varadarajan 1997)
- Fine analysis of the nature of component dependency
- Distinguishing \vec{X}_t and \vec{Y}_t in fixed dimension
- Further connections with p -adic physics and non-archimedean modeling
- Efficient simulation

Bibliography

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